

Cooperative Learning to Support Thinking, Reasoning, and Communicating in Mathematics

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A NEW DIRECTION

In considering how best to prepare students for the challenges of the next century, educators are changing the content of the mathematics curriculum and the ways we teach it. We are moving from a focus on arithmetic and computational skills toward a curriculum that develops students' abilities to think, reason, and communicate mathematically. The goal is to help students construct their conceptual understanding of mathematics, not just memorize facts and rules.

The teaching of mathematics is likewise changing in order to meet these new goals. Instead of teaching by telling or by demonstration, a blend of instructional methodologies is recommended that includes individual and group work and direct instruction. The focus is to provide frequent opportunities for students to explore and solve problems, individually and with others; and to develop their mathematical skills in the context of this exploration. The teacher is a facilitator of learning, guiding students' explorations, asking questions that extend their thinking, and encouraging students to communicate their thinking. Direct instruction is provided as the need emerges during this process.

The mathematics curriculum outlined by the National Council of Teachers of Mathematics (1989) suggests standards for kindergarten through grade twelve and lists content and strategies that should receive increased or decreased attention, such as the following:

Increased Attention	Decreased Attention
Number sense and meaning of operations	Complex paper and pencil computation
Students create own algorithms and develop own problem-solving strategies	One answer, one method, and rote memorization of rules
Writing and talking about mathematical thinking and strategies	Focus on the answer
Student use of manipulative materials	Worksheets
Use of a variety of instructional methods, such as small groups, individual explorations, peer instruction, whole- class discussions, project work	Teaching by telling; teacher and text as exclusive sources of knowledge
Assessing learning as part of instruction	Testing for sole purpose of assigning grades
Mathematics for all students	Mathematics for only some students

RATIONALE

Why does cooperative learning deserve a central place in mathematics instruction? The study of mathematics is often viewed as an isolated, individualistic, or competitive matter. One works alone and struggles to understand the material or solve the assigned problems. Perhaps it is not surprising that many students and adults are afraid of mathematics and

develop math avoidance or math anxiety. They often believe that only a few talented individuals can function successfully in the mathematical realm. Small-group cooperative learning addresses these problems in several ways.

- Small groups provide a social support mechanism for the learning of mathematics. “Small groups provide a forum in which students ask questions, discuss ideas, make mistakes, learn to listen to others’ ideas, offer constructive criticism, and summarize their discoveries in writing” (National Council of Teachers of Mathematics [NCTM], 1989, p. 79). Students learn by talking, listening, explaining, and thinking with others.
- Small-group cooperative learning offers opportunities for success for all students in mathematics (and in general). The group interaction is designed to help all members learn the concepts and problem-solving strategies.
- Mathematics problems are ideally suited for cooperative group discussion because they have solutions that can be objectively demonstrated. Students can persuade one another by the logic of their arguments. Mathematics problems can often be solved by several different approaches, and students in groups can discuss the merits of different proposed solutions.
- The field of mathematics is filled with exciting and challenging ideas that invite discussion. Mathematics offers many opportunities for creative thinking, for exploring open-ended situations, for making conjectures and testing them with data, for posing intriguing problems, and for solving non-routine problems. Students in groups can often handle challenging situations that are well beyond the capabilities of individuals at that developmental stage.

RESEARCH OUTCOMES

Reviews by Davidson (1985, 1990b) and Webb (1985) focused specially on mathematics learning, addressing achievement and group interaction and dynamics, respectively. Davidson (1990b) reviewed about eighty studies in mathematics comparing student achievement in cooperative learning versus whole-class traditional instruction. In over 40% of these studies, students in the small-group approaches significantly outscored the control students on individual mathematical performance measures. In only two studies did the control students perform better, and both of these studies had design irregularities. In support of these findings, Webb found a high correlation between students who gave directions in mixed-ability mathematics groups and their own achievement.

Davidson also found that the effects of cooperative learning of mathematical skills were consistently positive when there was a combination of individual accountability and some form of team goal or team recognition for commendable achievement. The effects of small-group learning were not significantly different from traditional instruction if the teacher had no prior experience in small-group learning, was not aware of well-established methods, and did little to foster group cooperation or interdependence.

For many mathematics teachers, the social benefits of cooperative learning are at least as important as the academic effects. In his review, Davidson concluded that cooperative learning is a powerful tool for increasing self-confidence as a learner and problem solver and for fostering true integration among diverse student populations.

A recent volume of the *Journal for Research in Mathematics Education* was devoted to cooperative learning in mathematics and included reports by several investigators (Davidson & Kroll, 1991; Dees, 1991; Yackel, Cobb, & Wood, 1991; Webb, 1991).

THE TEACHER’S ROLE

A cooperative mathematics lesson might begin with a meeting of the entire class to provide an overall perspective. This may include a teacher presentation of new material, class discussion, posing problems or questions for investigation, and clarifying directions for the group activities. The class then works in small groups, often in pairs in the elementary grades and groups of four in upper grades. Each group has its own working space, which in the upper grades might include a flip chart or section of the chalkboard. In their groups, students might work together to discuss mathematical ideas, solve problems, look for patterns and relationships in sets of data, and make and test conjectures. Students actively exchange ideas with one another and help each other understand their work. The teacher circulates from group to group, providing assistance and encouragement and asking thought-provoking questions.

Planning

As the teacher plans, several questions specific to cooperative mathematics lessons need to be addressed, including the following:

- What are the important mathematical concepts of this lesson? How will I help students link these goals with previous work and with long-term goals?
- Does the problem or exploration allow for multiple strategies, perspectives, and solutions?
- What opportunities for direct instruction or class discussion of a mathematics concept or skill might arise from students' exploration?
- What are some possible opportunities for supporting social, as well as mathematical, learning?
- How can I make this lesson interesting, accessible, and challenging for students at all levels of mathematical understanding and proficiency?
- How will this lesson provide opportunities for students to make decisions about such things as questions to explore and strategies and tools to use for problem solving?
- What open-ended questions might extend students' thinking? What questions might be asked to introduce the lesson? What questions might help students during group work? What questions might help students reflect on their experience?
- How can I link assessment with instruction?
- What are appropriate extension activities for groups that finish early or for the next day?

Introducing a Lesson

Cooperative mathematics lessons might begin with a problem statement or a question for exploration. Setting a context for the mathematical investigation serves as a motivator and helps students link the exploration to their own lives. It is also important to help students understand the mathematical goals of the lesson and how these goals connect to prior lessons and learning; what is expected of them; and how they will be held accountable. In addition, the teacher might facilitate a discussion about the group work by asking questions, such as "What do you want to find out?" "What are some possible strategies you might try?" "What has helped your group work well in the past?" "How will you share the work?" and "How will you agree on decisions?"

Facilitating Group Work

During group work, the teacher's role is to encourage students to define the problems they are investigating, to solve interpersonal problems, and to take responsibility for their learning and behavior. The teacher may intervene to refocus a group, to help its members see a problem from another perspective, to ask questions that extend mathematical and social learning, or to assess understanding. The teacher should try not to interrupt the flow

of the group work and, instead, should wait for a natural pause in the action. The teacher might ask open-ended questions that require progressively more thought or understanding and avoid giving the impression that there is a “right” answer. It is important to allow groups time to solve a problem themselves before intervention. Asking key questions to help them resolve the difficulty is more beneficial than solving the problem for them or giving lengthy explanations.

Helping Students Reflect

Reflection on the mathematical and social aspects of group work helps students develop their conceptual understanding as they discuss their experience and hear about the strategies, problems, and successes of others. Asking open-ended questions helps students consider such important issues as the following:

- What strategies did you try? Were they effective? Why?
- What solution do you think is reasonable? Why do you think that? Is there another solution that might work?
- What problems did you have? How did they affect the group? Were you able to solve them? If so, how?
- What are some ways to work that you would recommend to other groups?
- What was something someone in your group did that was particularly helpful to you or the group?

The reflection time should be kept relatively short and include use of a variety of methods such as group reflection, whole-class reflection, and reflection through writing or drawing pictures. If appropriate, the teacher might wish to give feedback regarding both successes and problems he or she observed as groups worked.

Figure 15.1
Example of a Class Chart

What is the most important quality in a friend? Make a tally mark

Honesty	111
Loyalty	I
Cheerfulness	II
Good Listener	1111
Intelligence	
Dependability	<i>111</i>

COOPERATIVE LEARNING FOR ELEMENTARY MATHEMATICS INSTRUCTION

Cooperative learning strategies can be easily integrated into the elementary mathematics classroom. The choice of strategy and group size will depend on the time of year, the sophistication of students with cooperative problem solving, and the mathematical activity itself.

Individual Activities That Lead to Cooperative Discussion

Graphing

After students have individually responded to a class graph, chart, or table such as Figure 15.1 (Freeman, 1986), several simple cooperative strategies can encourage students to analyze and discuss the data.

Simple Cooperative Strategies

Turn to Your Partner: Students turn to a person sitting next to them to discuss an issue or question, such as “What do you notice about the data?”

Heads Together: Similar to Turn to Your Partner, but for groups of four. Students might put their heads together, discuss the data, and agree on at least two factual mathematical statements, such as “More than half the students chose honesty and dependability,” or “The same number of students chose honesty as good listener.” These statements can then be discussed as a class.

Think-Pair-Share: Students individually take a short period of time to think about a question such as “What inferences can you make from the data?” and discuss their thoughts with a partner. (An example of an inference statement might be “Intelligence is not an important quality in a friend.”) Pairs then share their thinking with another pair (**Think-Pair-Square**) or report their thinking to the class.

Think-Pair-Write: This structure is like Think-Pair-Share, except that pairs write about their thinking after they have discussed their thoughts with each other. Pairs can exchange their written statements and decide whether or not they agree with the other pair’s statement. The teacher might also collect the statements and place them near the graph for students to read, discuss, and decide whether or not they agree with the statement (see Kagan, 1992).

Mental Math

One way to encourage student discussion of mental strategies is to ask them to individually think for a short time about a question, such as “How many ways can you double 38 mentally?” Groups of four can then discuss and list as many strategies as they can and make sure each member can explain one of the strategies. The teacher then asks the class to stand and has one student explain a strategy. After checking with the class for understanding and agreement, the teacher asks all students with a similar strategy to sit down. The activity continues until all students are sitting.

Paired Activities

Pairs are an important group size for elementary mathematics lessons. In a pair, both students can easily manipulate materials and have frequent opportunities to share their thinking.

Games

Many mathematical games can be turned from a competitive experience to a cooperative one. The game of POISON! (Burns 1991b) is a good example. The rules for two players are as follows:

- You need thirteen objects.
- Take turns. On your turn, take away one or two of the objects.
- The last object to be taken is POISON! Whoever gets stuck with it loses.

To make this a cooperative activity, you might add these directions:

- Your job as a pair is to discover a winning strategy for this game.
- Play several games. Discuss what you notice about the game.
 - When you think that your pair has a winning strategy, play with another pair or with the teacher.
- As a pair, discuss what you learned.

For an extension activity, students can develop strategies for playing this game with a number of objects other than thirteen.

Paired Practice

It is now recognized that mathematical skills are most effectively developed in the context of their use. Some practice, however, is appropriate and can be a paired activity.

Guidelines for pairs completing a practice page in their textbook might be as follows:

- Work together to solve problems.
 - Be sure you both agree to the solution before you go on to the next problem.
 - Be sure you can both explain how to get your solutions.
 - Both sign your record sheet to show you have checked and agree to the solutions.
- An extension of this activity might be for pairs to exchange and check each others' record sheets:
- Exchange your record sheet with another pair.
 - Check the solutions. If you both agree, sign the record sheet.
 - If you disagree with a solution, write about why you disagree or discuss the problem with the other pair.

Partner Interview

One way to conduct a partner interview is for one partner to interview the other, then switch roles. Pairs can then discuss points of agreement and disagreement, write about their thinking, and/or share their thinking with other pairs or the class. Examples of interview questions might include the following:

- The weather person said that there is a 100% chance of rain for tomorrow. Is this a good prediction for this month? Why?
- Tanika loaned Jamal a dollar. She said that the interest would be 75% a day. Is this a good deal for Jamal? Why?

Partner Problem Solving

Students benefit from solving mathematics problems individually and with others. Some guidelines that might help pairs or groups solve problems include the following:

- Tell each other the problem and discuss anything you don't understand.
- Talk about different ways you could solve the problem.
- Decide on a strategy or strategies to try.
- Try your strategy or strategies. Discuss the results. Try other strategies if you wish. Decide on a way to show what you did and to share your results.

Partner problem solving is particularly appropriate for open-ended problems that provide opportunities for students to have many interpretations, strategies, and solutions. For example: "Your class of thirty-two students is going on a picnic. How many packages of hot dogs and how many packages of buns will we need?" Problems such as this one can

lead to exciting discussions not only about strategies and solutions, but also about assumptions students made in order to solve the problem, such as the number of hot dogs and number of buns in a package, how many hot dogs each student would eat, whether some students would not eat any hot dogs, how many people would be going on the picnic, and many others.

Menus

Menu activities are a set of activities or problems that provide numerous opportunities for students to explore a set of mathematical concepts. The teacher introduces approximately four to eight related activities that may be the work for several days, a week, or several weeks. Students, in pairs or groups, are responsible for choosing and completing a specified number of the activities during the time period. This provides opportunities for students to make decisions and learn to manage their time. During or at the end of the time period, each activity is discussed in terms of students' approaches to the problem and solutions. (For an in-depth discussion of how to use menus and examples of menus for elementary grades, see Burns, [1991bJ].)

The following three activities might be part of a menu focusing on multiplication and division.

Hit the Target

You need: A partner, a calculator, your journal.

The game: Starting at 36, hit the target range of 2,000—2,100, using as few guesses as possible.

How to play:

1. One partner enters 36 into the calculator and presses x.
2. The other partner estimates a number that when multiplied by 36 will produce a product in the target range.
3. If this estimate misses, the first partner must estimate a number to multiply with the number now showing on the calculator window to produce a product in the target range.
4. Keep playing, taking turns and keeping track of guesses, until you hit the target range (NCTM, 1991).

Mystery Equation

You need: A partner, your journal.

To do: Copy the following in your journal:

$$\begin{array}{r} \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}} \\ \times \hspace{1cm} \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}} \\ \hline \end{array}$$

Choose any five digits, discuss each of these questions, then write about your thinking.

1. How would you arrange the digits in the problem to produce the largest product possible?
2. How would you arrange the digits in the problem to produce the smallest product possible?
3. How many different products are possible using the five digits you have selected?

Choose another five digits and answer the questions again. Suggest a method for arranging any five digits as a three-by-two-digit multiplication problem to form the largest and

smallest possible product (NCTM, 1991).

What Happens When?

You need: A partner, your journal.

To do: Find out what happens when you multiply a number less than 1.

Copy the table below into your journal and complete it. Discuss the following questions and write about your thinking.

Pick a whole number	Multiply by 0.05	Multiply by 0.48	Multiply by 0.9

1. In general, what happens when you multiply a whole number by 0.05?
2. In general, what happens when you multiply a whole number by 0.48?
3. In general, what happens when you multiply a whole number by 0.9? (NCTM, 1991)

Group Activities

Many cooperative elementary mathematics activities are appropriate for groups of three or four students.

Many Pieces Problems

Many Pieces problems are a good introduction to group problem solving, as each student has specific information to contribute to the group. Each group receives clue cards that are then dealt one to a group member. Students share their clues verbally, without showing each other their clue cards, and work together to solve the problem. (Some forms of these problems have two extra clues that are put face down and turned over when the group needs another clue or wants to check to see if its solution fits with all the clues.)

Many Pieces problems can be developed from problems found in a textbook. A typical problem might be the following: Maria is learning to swim. She swam one lap on the first day, three laps on the second day, six laps on the third day, and ten laps on the fourth day. If this pattern continues, how many laps will she swim on the fifth day? This problem can be written as four clues:

- Maria is learning to swim. She swam one lap on the first day.
- Maria swam three laps on the second day.
- Maria swam six laps on the third day.
- Maria swam ten laps on the fourth day. How many laps will she swim on the fifth day?

(See EQUALS, 1989, and GEMS, 1992, for Many Pieces problems with a focus on logical thinking.)

Group Problem Solving

Short- and long-term group explorations, investigations, and problem solving activities provide opportunities for students to cooperatively tackle problems that are not easily

solved, to define questions to be studied, to determine the information required, to decide on methods for obtaining this information, and to determine the limits of acceptable solutions. These experiences might include real-world investigations that involve students in solving actual problems that relate to their lives, for example, studying food waste at lunchtime and making recommendations to the principal based on their research. These experiences might be projects that engage students in mathematical exploration and thinking as they develop a product, such as designing and making a quilt. These experiences might also involve students in investigating mathematical ideas and relationships, such as generating their own algorithms and developing ways to compare the capacity of several containers or the areas of various shapes. (See Robertson, Regan, Freeman, & Contestable, 1993, as a source for group problem-solving curriculum materials.)

Whole-Class Activities

Whole-class activities provide opportunities for classbuilding as well as for mathematical development. The following are just two of many possibilities:

Line-up Activities

Students line up based on a specific characteristic, such as their birth- dates. Many options for mathematical activities can follow. For example, the class can form a human birthday graph with all students born in January standing in a line, all born in February forming a second line, and so on. This produces a bar graph that can be transferred to paper. Students can also find the median birthdate and the percentage or fraction of the students born in each month.

Four Corners

Students with a common characteristic meet in a corner of a room. In a primary classroom, for example, corners might be labeled with numbers such as 0, 10, 20, and 30. Students go to the corner with the number closest to the day of the month they were born. The teacher asks questions that help students explore the relative magnitude of numbers. The numbers might then be changed to 5, 15, 25, and 35. Students move to the corner now labeled with the number closest to the day of the month they were born. Students discuss how their position and the groups have changed, as well as issues of relative magnitude.

COOPERATIVE LEARNING FOR SECONDARY MATHEMATICS INSTRUCTION

To get started using cooperative learning in the secondary mathematics classroom, the teacher might begin with some simple cooperative structures that encourage student interaction. The following describes several useful structures and sample applications for use in general mathematics, algebra I, and geometry classes.

Simple Cooperative Structures

The Three-Step Interview

The Three-Step Interview is conducted in a group that consists of four members subdivided into pairs: A and B, C and D.

Step one: A interviews B while C interviews D. The interviewer listens, asks questions, and paraphrases, but does not elaborate or share personal data.

Step two: Reverse roles: B interviews A, while D interviews C.

Step three: Share-around: Each person shares information about his or her partner in the group of four.

Math Class Applications of Three-Step Interview

General math: How can you use fractions? What examples of fractions did you find in the newspaper?

Algebra I: What are some variable attributes of people in our class or of our classroom (for example, height or age)?

Geometry: Give examples of geometric shapes that occur in your home.

Thoughtful responses to interview questions are more likely to occur with advance preparation by the students. It is useful to give these interview questions to the students a day or two in advance.

Roundtable

The teacher poses a question having multiple answers or gives each group a worksheet. The group has only one piece of paper or worksheet and perhaps only one pencil. One student in the group writes down one response, says it aloud, and then passes the paper or worksheet to the person on his or her left. The process continues in this way. A student may choose to pass on one round and give an answer the next time.

Math Class Applications of Roundtable

General math: Find combinations of three numbers whose sum is 60.

Algebra I: Give specific examples of the rule $(a'')' = a'''$.

Geometry: State a variety of terms in geometry (for example, “point”).

Roundtable can be used for creative brainstorming, simple applications, or review. It fosters cooperative skills such as taking turns and listening without interrupting.

Numbered Heads Together

Each student in the group is given a number from one to four. The teacher poses a question, issue, or problem. Students talk this over within the group and prepare to respond. The teacher then calls upon students by number to represent the group.

Math Class: Applications of Numbered Heads Together

General math: Find all factor pairs for a given whole number.

Algebra I: State examples of perfect square trinomials.

Geometry: State possible values for the base and height of a set of rectangles that all have the same given area.

Because Numbered Heads Together allows discussion and interaction, it is more appropriate than Roundtable for more difficult mathematical questions. Numbered Heads Together gives all students equal opportunities to respond successfully in the whole class after small-group discussion.

Think-Pair-Share

As mentioned earlier in this chapter, this strategy allows the teacher to pose questions to the students sitting in pairs. Students silently think of a response individually for a given period of time, then pair with their partners to discuss the question and reach consensus. The teacher then asks students to share their agreed-upon answers with the rest of the class.

Math Class Applications of Think-Pair-Share

General math: Describe the meaning of the terms area and perimeter and show how to

compute them for a rectangle.

Algebra I: Is the slope computed for a given line always the same, no matter which two points on that line are used to calculate that slope? Illustrate with examples.

Geometry: Clarify the relationships among these figures: rectangle, square, rhombus, parallelogram.

Think-Pair-Share can be used for discussion of concepts or procedures, for problem solving, or for practice. It can also be used spontaneously to clear up ideas given during a presentation. The teacher asks the students to think about what has been presented, and then, in discussion with a partner, students either clarify it or pose a question. Clarifications or questions are then stated to the whole class.

Here is an example of a guess-my-rule task well suited for Think-Pair-Share. The teacher asks students working first individually and then with a partner to find possible rules for the following function tables:

x f(x)	x g(x)
0 2	0 1
1 5	1 2
2 8	2 5
3 11	3 10
4 14	4 17
x h(x)	x k(x)
0 2	0 2
1 3	1 4
2 10	2 8
3 29	3 16
4 66	4 32

Pairs Check

The Pairs Check procedure is used for practice and mastery of skills or procedures. Within groups of four, each pair takes two roles: performer (or solver) and coach.

1. One partner performs the task or solves the problem. The second partner functions as coach by observing carefully, giving hints or pointing out errors as needed, and giving positive feedback to the first partner. (Exaggerated praise can be used here for fun.)
2. Partners switch roles for the second performance or problem in the set.
3. The two pairs check their responses to the first two problems to see if they agree. When both pairs agree, they give a team handshake or cheer.

Math Class Applications of Pairs Check

General math: Find the greatest common factor of two whole numbers.

Algebra I: Complete the square for a quadratic expression.

Geometry: Construct the perpendicular bisector of a line segment.

Group Problem Solving

Once students have become skillful in working in pairs or groups with simple structures, they may be ready to solve some challenging problems in pairs or groups of four. The teacher then states guidelines, such as the following, to foster cooperative group problem solving:

1. Cooperate with other group members.
2. Achieve a group solution for each problem.
3. Make sure that everyone understands the solution before the group goes on.
4. Listen carefully to others and try, whenever possible, to build upon their ideas.
5. Share the leadership of the group.
6. Make sure that everyone participates and no one dominates.
7. Take turns writing problem solutions on the board.
8. Proceed at a pace that is comfortable for your own group.

The following is an appropriate problem for pairs in algebra or geometry. The teacher asks the students to work together to solve the problems and to make sure both partners understand their strategies and solutions.

1. a. Find the area of the rectangle with vertices at $(-4, 5)$, $(3, -1)$, and $(-4, -1)$.
b. Find the coordinates of the fourth vertex.
2. Refer to problem 1. In each case, first predict how the area of the rectangle will change. Then graph and check your prediction.
 - a. The x- and y-coordinates of each vertex are multiplied by 2.
 - b. The x- and y-coordinates of each vertex are decreased by 1.
 - c. The x- and y-coordinates of each vertex are replaced by the opposites of the original vertex.

Group problem solving can be used in the discovery and/or proof of mathematical generalizations. For example, in algebra¹, students in groups can readily discover laws of exponents such as $b^m \cdot b^n = b^{m+n}$. They do so by choosing a value for b and systematically selecting values for m and n , recording results, and looking for a pattern.

b	m	n	b^m	b^n	$b^m \cdot b^n$	$=$	b^r
7	2	3	7·7	7·7·7	7·7·7·7·7	=	7 ⁵
7	3	4	_____	_____	_____	=	_____

Groups can also pose problems for other groups to solve. This is sometimes known as “send a problem” or “pass a problem.” For example, each group might pose a coin problem such as the one that follows. Such problems can lead to lively discussions between groups, because there is often more than one correct solution that fits the stated conditions. (Incidentally, educated “guess and test” procedures are more manageable than algebra for these coin problems.)

A person has nineteen coins, which may include nickels, dimes, and quarters. The total value of the coins is \$2.95. Find the possible number of coins of each type. Is there a unique answer? It may help you to use a data table.

Q	D	N	# of Coins	Total Value
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Many Pieces

The Many Pieces problem described earlier in this chapter is also a useful structure for group problem solving in secondary mathematics. Consider the following problem one might find in a textbook:

The Simpson's are converting a section of a warehouse into an apartment. They are decorating an open 20' x 30' area that will serve as both living room and dining room. They have bought a new 9' x 12' carpet for the living room area. Mrs. Simpson's mother gave them an antique braided circular rug, 10' in diameter, that they plan to use in the dining area. The Simpson's will paint only the floor area not covered by the carpets. How much floor area needs to be painted? After some editing, this case, converted to the Many Pieces format, gives us five clues:

- The Simpson's are converting a section of a warehouse into an apartment.
- Mrs. Simpson's mother gave her an antique braided circular rug, 10' in diameter that she plans to use in the dining area.
- The Simpson's have bought a new 9' x 12' carpet for the living room area.
- The Simpson's will paint only the floor area not covered by the carpets.
- They are decorating an open 20' x 30' area that will serve as both living room and dining room.

You might wish to make a scale drawing of a possible layout on graph paper.

Jigsaw

Jigsaw is another useful structure for cooperative problem solving. The following are guidelines for group problem solving using Jigsaw.

1. Task division: A task or passage of text material or a problem set is divided into several component parts (or topics).
2. Home groups: Each group member is given a topic on which to become an expert.
3. Expert groups: Students who have the same topics meet in expert groups to discuss the topics, master them, and plan how to teach them.
4. Home groups: Students return to their original groups and teach what they have learned to their group members.

Math Class Applications of Jigsaw

General math: Compute the area and perimeter for various triangles.

Algebra I: Explore slopes of lines.

Geometry: Prove congruence of pairs of triangles, using SSS, SAS, ASA, or base and adjacent angle in an isosceles triangle.

The procedure for exploring slopes of lines is as follows: In your expert group, make a table of values and carefully draw the graph of each assigned line. Put all your graphs on the same axes. Be prepared to explain your graphs to others.

1. $y = x$, $y = 2x$, $y = 3x$.
2. $y = 1/2x$, $y = 1/3x$, $y = 1/4x$.
3. $y = -x$, $y = -2x$, $y = -3x$.
4. $y = -1/2x$, $y = -1/3x$, $y = -1/4x$.

Discuss the effect on the graph of changing the coefficient of x in the equation $y = mx$. (The number m , the coefficient of x , is called the *slope* of the line. The terminology can be introduced after the activity demonstrating the need for it.)

A similar Jigsaw activity varies the y -intercept while keeping the slope constant. In other words, for the equation $y = mx + b$, we fix m , vary b , and notice the effect on the graphs.

Group Projects and Investigations

Group projects and investigations often require several days to several weeks and may require groups to work

outside of class. Groups gather and analyze data as steps toward problem solving and decision making, then prepare a report for presentation to the class. Many projects can be conducted using the Group Investigation model (see chapter 7 in this volume).

A textbook problem is the genesis of a small-group investigation for general math students. This problem concerns a rate and a range for a particular car and reads: A particular car is advertised as getting 25 to 32 miles per gallon. If the gas tank holds 11.4 gallons, what is the range for the number of miles the car can travel on one tank of gasoline?

After class discussion of this problem, ask groups to discuss the following question, which also incorporates the mathematical concepts of maximum and minimum, and ways they might gather data: Of the cars sold today, which have the shortest and which have the longest range?

As a class, discuss the question and ideas groups have for collecting data. Have groups decide how to divide the data collection. (To make the task manageable, groups may decide to research representative cars in several categories.) Be sure groups know how much class time they will have and the due date for their presentation of their findings. Provide time for groups to meet to plan their work.

When groups are ready, have them present their data to the class. First in groups, then as a class, compare the data from all the groups and see if consensus can be reached regarding the cars with the longest and shortest ranges.

Another example of a group project for secondary students is to investigate the variables that impact how high a ball bounces. For this project, each group will need two meter sticks, taped one on top of the other, and one tennis ball or superball per group. Begin by posing the question and asking groups to discuss the variables they think can affect how high a ball bounces. (Groups may brainstorm such variables as type of ball, surface, initial velocity, and so on.)

Begin by asking groups to test the effect of dropping balls from different heights. You might wish to suggest that groups assign members these roles: someone to drop the ball, two observers, someone to record the original height H and bounce height B , and someone to compute. Assign different heights H from which to drop the ball: 200 cm, 190 cm, and so on. Ask each group to drop the ball five times from its own height H . Have each group compute the average B of its five bounce heights and the ratio B/H .

Record the data from all groups on the board and ask groups to look for patterns. Students are astonished to find that the ratio B/H is a constant for a given type of ball and surface. (Students in more advanced courses note that the ratios for consecutive bounces—first, second, third, and soon—are terms in a geometric progression, for example .6, (.6)², (.6)³, and so on.) Ask groups to propose various modifications of the experiment to further test this finding as well as to test other variables; then have them experiment.

The following are examples of other group projects:

1. Design a city park including a playground within a specified cost range (Johnson & Johnson, 1991).
2. Plan a talent show, using a computer simulation to keep track of the effects of variables such as the ticket price, costs of publicity, costs of refreshments, and other expenses (Sheets & Heid, 1990).
3. Design a theater according to specifications about the shape, number of seats, and other variables (Serra, 1989).
4. Construct geometric solids such as prisms, pyramids, or dodecahedra.
5. Conduct survey research, for example, about political preferences or consumer choices.
6. Observe and record traffic flow at a busy intersection (with carefully regulated safety precautions).
7. Gather and analyze environmental data, for example, relating to the proportion of certain substances in different water samples.

Cooperative learning is a viable and effective instructional methodology for teaching and learning mathematics and helps make mathematics exciting and enjoyable for both students and teachers. Cooperative strategies can be integrated at any grade level and for any mathematical topic. Davidson (1990a) reported that many positive effects are noted by teachers and students. Students “learn to cooperate with others and to communicate in the language of mathematics. The classroom atmosphere tends to be relaxed and informal, help is readily available, questions are freely asked and answered, and even the shy student finds it easy to be involved. Students tend to become friends with their group members, and the teacher- student relationship tends to be more relaxed.” In addition, “Many students maintain a high level of interest in the mathematical activities” and “have an opportunity to pursue the more challenging and creative aspects of mathematics while they achieve at least as much information and skill as in more traditional approaches.”

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Co-operative Learning and Small Group Learning and Mathematics

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In many countries, teachers of mathematics are being encouraged to move from presentation-recitation modes of teaching towards a blend of instructional methods. An increasing emphasis on real problem-solving, investigations, projects and other forms of applying mathematical knowledge and skills in everyday life situations is changing the nature of mathematics in schools. There has been a growing demand from both professional and business people and education policy-makers to stimulate active learning, promote effective teaching, and encourage appropriate

assessment methods to be utilised in the teaching and learning of mathematics. Active learning has been interpreted to mean more participatory learning methods, such as communication, co-operation and working on real-life problems. Therefore, many teachers and school improvement experts seeking for better quality teaching and learning have turned their eyes to small group learning methods that have been developed since the late 1960s.

The overall objective of this sub-area of the Centre's research focuses on small group learning in mathematics as a method of raising the quality of teaching and learning, and as a consequence pupil achievement in schools worldwide.

Restructuring whole classes into small groups of pupils has been seen as one pedagogical response to these challenges. In this research we use the term co-operative learning to refer to dividing a large group into groups of two to six pupils and assigning specific tasks to these groups. However, this kind of general definition may be confusing because there are in fact several different models of co-operative learning that vary considerably by their epistemological orientations and practical implications for the roles of the teacher and the learners. Particularly in mathematics, cooperative work can be used in conjunction with practising skills, doing investigations, collecting data, discussing concepts and principles, or solving mathematical problems.

Choosing the task for teaching and learning a topic in mathematics influences what and how we expect the students to learn that topic. Yet relatively little attention has been given to the role and nature of learning tasks in mathematics that promote high-level interaction and productive co-operation between team members of a small group. The relationship between the curriculum, teaching and learning is a complicated one. The curriculum should be seen as more than a collection of facts, rules, algorithms and a collection of activities to deliver them and assess them. It must be a coherent view of mathematics that develops connections between the multiple representations of each topic. Teaching is most effective when the teacher understands what the learner knows and has the tools, tasks and techniques to challenge and support the learner to move forward by learning mathematics with understanding. In a paper (*One and One is sometimes Three in Small Group Mathematics Learning*) published in the *Asia Pacific Journal of Education* (22 No 1, 2002), John Berry and Pasi Sahlberg propose a typology of tasks in school mathematics in which the importance of the role of the task in teaching and learning mathematics is discussed. We propose a classification of mathematics tasks linked to the good ingredients of teaching and learning mathematics and relate the concept of equal exchange models to this typology of tasks.

A recent study on the use of small group learning in England and in Finland has been carried out by John Berry and Pasi Sahlberg. For a copy of the research report please contact the authors. Further work on co-operative learning and Mathematics is continuing.

Publications

Sahlberg P. and Berry J., *Small Group Learning in School Mathematics: A study on teachers' and pupils ideas about groupwork in mathematics in England and Finland, (in press)*.

The style of teaching and learning mathematics and the type of tasks used affect the conceptions, attitudes and views of mathematics held by teachers and pupils. Our purpose in this research study is to investigate the occurrence of small group learning and the methods of co-operative learning in school mathematics in Finland and in England. Furthermore we wish to draw more educators' and researchers' attention to the role of these methods of learning and to the importance of task design in implementing them into school mathematics. Teaching mathematics in school is a challenging activity and the role of the task is often overlooked by educators and teachers. As part of the research study we propose a typology of group tasks in mathematics that uses the notion of

exchange models.

We conclude from our research that (i) small group learning is not common or implemented very often in the teaching and learning of mathematics in Finland or England; (ii) teachers believe that the role of pupils working in small groups is to develop social and communication skills and for doing mathematics not for learning new mathematical concepts and skills; (iii) the tasks used in school mathematics determine the teaching and learning styles and these are not conducive to the methods of co-operative learning.

Sahlberg P. and Berry J., 2002, One and One is Sometimes Three in Small Group Mathematics Learning, *Asia Pacific Journal of Education*, Vol 22 No 1, pp 83 - 94

In recent years, Mathematics teaching has been confronted by demands for higher standards and better pupil achievement in several parts of the world. Researchers have suggested a shift from teacher-centred instruction towards more active participatory learning methods as one way to improve the quality of the learning process. The tension between whole class teaching versus small group learning in Mathematics has been particularly apparent in many education systems. This article analyses the development of Mathematics teaching by asking whether small group learning is an effective arrangement in teaching school Mathematics. We conclude that although there is no unanimity about the effects of small group learning on student achievement in school Mathematics, it seems that it produces at least equal academic outcomes among all students compared to more traditional methods of instruction. Working in pairs is a particularly effective form of learning Mathematics and that small groups are beneficial for developing mathematical problem-solving skills. We also conclude that the present educational policies and increased quality assurance structures in many countries conflict, or are not consistent with scientific-professional thinking and research on the teaching of mathematics.

Berry J. and Nyman M., Small Group Assessment Methods in Mathematics, *The International Journal of Mathematics Education in Science and Technology*, Vol 33 No 5, pp 641-649, 2002

Assessment of students' attainment in courses is often driven by the method of instruction. When mathematics is taught in the traditional style of lectures on theory coordinated with homework on standard problems, the testing is often oriented to reproducing the skills demonstrated by the instructor. If a more collaborative teaching method is used, how does one assess the students' acquisition of problem-solving and mathematical thinking skills. In this paper we discuss a team-oriented formal testing method used in a mathematical modelling course taught during the Alma College intensive spring term.

Sahlberg P. and Berry J., 2002, Matematiikan oppiminen pienryhmissä (Small Group Learning in Mathematics) in eds. Sahlberg, P. and Sharan, S. *Yhteis-toiminnallisen oppimisen käsikirja (Handbook of Co-operative Learning Methods)* Chapter 2.1, pp 176-198, WSOY: Helsinki, Finland. ISBN 951-0-25962-4

Graham E., Rowlands S., Jennings S. and English J., 1999. Towards Whole-class Interactive Teaching. *J Teaching Mathematics & its Applications*, Vol. 18 No 2, pp 50-60

This paper considers an approach to addressing the decline in the level of achievement of British pupils in mathematics. It looks in detail at the differences between the teaching methods of Britain and Hungary, as research studies have indicated the high level of achievement of Hungarian pupils in mathematics. The paper outlines three theoretical perspectives (radical constructivism, social constructivism, and Vygotsky's zone of proximal development) that are helpful in considering the important differences. The major differences are considered under four categories: expectation and consistency, assessment, continuity and differentiated teaching. The paper proposes the method of whole-class interactive teaching as a way forward that would improve pupils' achievement, and gives

practical suggestions for developing such a teaching strategy.